Math 347: Homework 4
Due on: Oct. 8, 2018

1. Let $f : A \to B$ be a function and C a subset of A. Then one can define a function called the *restriction of f to C* denoted by

$$\begin{aligned} f|_C : C &\to B \\ c &\mapsto f(c). \end{aligned}$$

   (a) Prove that there always exist a (non-empty) subset $C \subset A$ such that $f|_C$ is injective.

   (b) Let C and D be subsets of A. Suppose that for two functions $f : A \to B$ and $g : A \to B$ one has

$$f|_C = g|_C \quad \text{and} \quad f|_D = g|_D .$$

   Is it true that $f = g$? Find a condition on C and D that guarantees that this is true.

   (c) Let C and D be subsets of A. Suppose that $f : C \to B$ and $g : D \to B$ are two functions. When is there a function $h : A \to B$, such that $h|_C = f\ \ h|_D = g$?

2. Let S be the set of differentiable functions $f : \mathbb{R} \to \mathbb{R}$. One has a function $D : S \to S$ given by $D(f) = f'$, i.e. the derivative of the function f.

   (a) prove that D is not injective;

   (b) prove that D is surjective;

   (c) find the largest subset $T \subseteq S$ such that the restriction of D to T is injective.

3. For any set S prove that there is a bijection between $P(S)$ and the set

$$T = \{f : S \to \{0, 1\}\},$$

   i.e. the set of functions from S to $\{0, 1\}$. What can you conclude about the cardinality of S and $P(S)$?

4. Let $S = \{1/n \mid n \geq 2\}$, and $T = S \cup \{0\} \cup \{1\}$, and consider the function $g : T \to S$ defined by

$$g(0) = 1/2, \ \ g(1) = 1/3 \ \ \text{and } g(1/n) = \frac{1}{n+2} \quad \text{for } n \geq 2.$$

   (i) Prove that g is a bijection;

   (ii) Use g to prove that $[0, 1]$ and $(0, 1)$ have the same cardinality[1];

   (iii) Prove that $\mathbb{R}$ and (a, b) have the same cardinality for all $a, b \in \mathbb{R}$ such that $a < b$.

5. Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by

$$f(n, m) = 2^{m-1}(2n - 1).$$

   Check that this function shows that $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ have the same cardinality. (Along your proof you might need an auxiliary fact, state and prove it. Consider $\mathbb{N} = \{1, 2, \ldots\}$ here.)

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[1] Notice that $[0, 1] = T \cup ([0, 1] \setminus T)$, and that $(0, 1) = S \cup ((0, 1) \setminus S)$.